1060 – Class 22
Fall 2013

2.5 – Inverse Functions

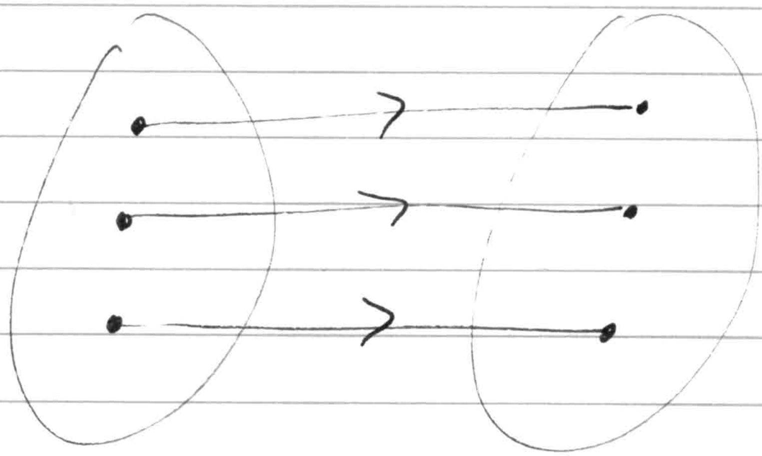
A common type of question is
let $f(x) = 2x+1$
find x s.t. $f(x) = 7$

$$2x+1 = 7$$
$$2x = 6$$
$$x = 3$$

3 is the # s.t. $f(3) = 7$

(To define "inverse function"
remember the definition of "function")

A function maps each element
of a set A
to a single element
of a set B



$$A \xrightarrow{f} B$$

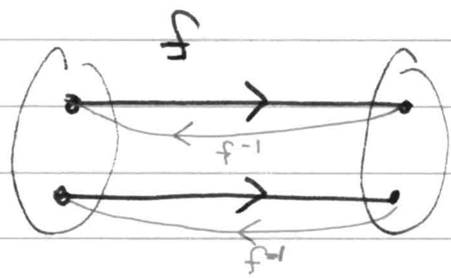
domain of f   range of f

Notice: flipping the arrows "undoes f"

If there is a <u>function</u> that undoes f
we call it the inverse of f
and write it f^{-1}

<u>Some functions have inverses:</u>

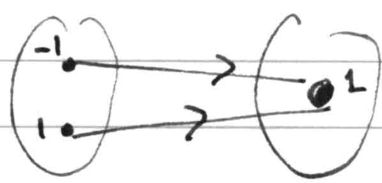
<u>Eg:</u>



flipping each arrow
defines [redacted] the inverse f^{-1} of f

<u>Not all functions have inverses</u>

<u>Eg:</u> let $f(x) = x^2$
then $f(-1) = 1 = f(1)$
so



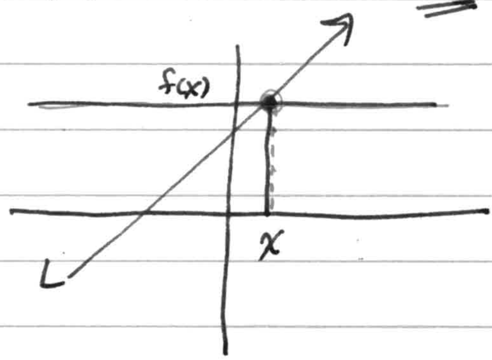
f [redacted] is a function,
but flipping the arrows
sends ONE input to TWO outputs

Define: f is one-to-one
if f never takes the same value twice

That is: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

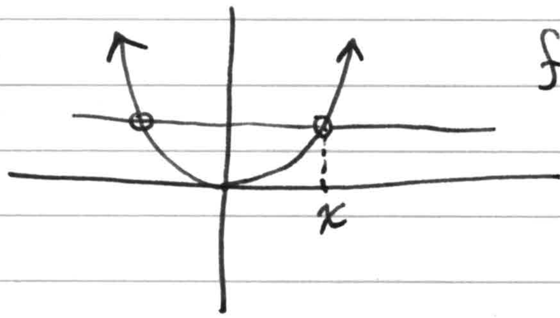
In pictures

Eg: $f(x) = 2x+1$ is one-to-one



for each x
No other input
outputs $f(x)$

Eg:



$f(x) = x^2$ is NOT 1-1

Two inputs output the same $f(x)$

Horizontal line test: f is one-to-one

$\Leftrightarrow$ No horizontal line intersects its graph
more than once

Now that we know which types of functions will have inverses, we can define

Define: If $f(x)$ is one-to-one

$$f^{-1}(y) = \left[\begin{array}{l} \text{the } \# \, x \\ \text{s.t. } f(x) = y \end{array}\right]$$

The idea is clean enough:

$$f(1) = 5 \Rightarrow f^{-1}(5) = 1$$

$$f^{-1}(17) = 2 \Rightarrow f(2) = 17$$

That is: f and f^{-1} undo eachother

In general,

$$\boxed{\begin{array}{ll} \text{if} & f(x_1) = y_1 \\ \text{then} & x_1 = f^{-1}(y_1) \end{array}}$$

So
$$f^{-1}(f(x_1)) = f^{-1}(y_1) = x_1$$
AND
$$f(f^{-1}(y_1)) = f(x_1) = y_1$$

$\Big\}$ f and f^{-1} undo eachother.

Eg: If $f(1) = 2$
$f(2) = 3$
$f(3) = 1$

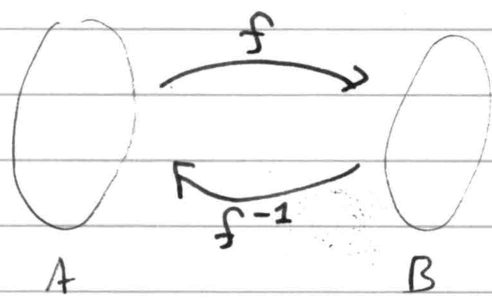
Then $f^{-1}(3) = \left[\begin{array}{l}\text{the \# } x \\ \text{s.t. } f(x) = 3\end{array}\right]$

so $f^{-1}(3) = 2$

AND $f^{-1}(2) = \left[\begin{array}{l}\text{the \# } x \\ \text{s.t. } f(x) = 2\end{array}\right]$

so $f^{-1}(2) = 1$

Because we flip the role of inputs & outputs



the domain of f^{-1} = the range of f

and

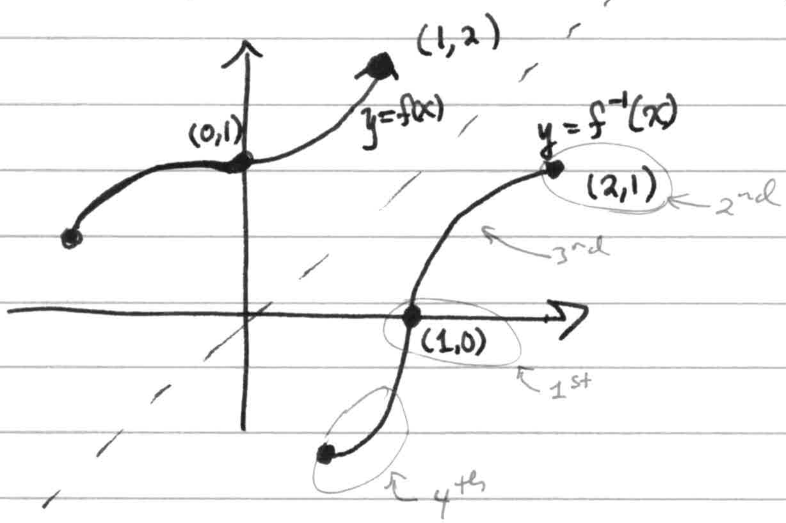
the range of f^{-1} = the domain of f.

the idea that we are switching the roles of inputs and outputs also tells us how

[redacted]

To find an inverse from a graph:

reflect across the line $y = x$



Often (not always) we can [redacted]
find an equation for $f^{-1}(x)$

Eg: $\boxed{\text{find the inverse to } f(x) = 3x^3+1}$ ← Question

the work.

Notice $y = 3x^3+1$ ← replace $f(x)$ with y

$\Leftrightarrow$

$$y-1 = 3x^3$$

$$\frac{y-1}{3} = x^3$$

$$\sqrt[3]{\frac{y-1}{3}} = x$$

} solve for x in terms of y

conclude: to output y
input $x = \sqrt[3]{\frac{y-1}{3}}$

$$\boxed{f^{-1}(y) = [redacted]\sqrt[3]{\frac{y-1}{3}}}$$ ← the answer

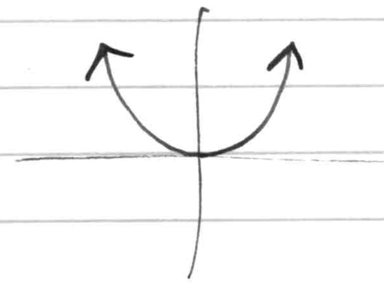
we can check that f & f^{-1} undo eachother

$$f\left(f^{-1}(y)\right) = f\left(\sqrt[3]{\frac{y-1}{3}}\right) = 3\left(\sqrt[3]{\frac{y-1}{3}}\right)^3+1 = y-1+1 = y \checkmark$$

$$f^{-1}(f(x)) = f^{-1}\left(3x^3+1\right) = \sqrt[3]{\frac{(3x^3+1)-1}{3}} = \sqrt[3]{\frac{3x^3}{3}} = x \checkmark$$

"Partial Inverses" – inverting part of a function

Notice: $f(x) = x^2$ is NOT 1-1



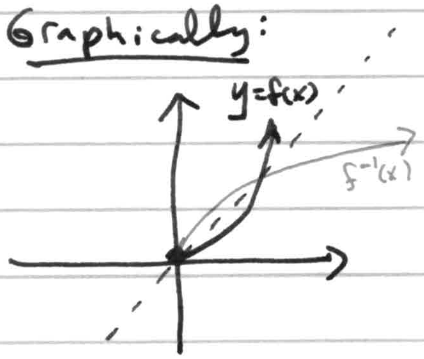
But if we cut off the left side by restricting to $x \geq 0$

$f(x) = x^2$ for $x \geq 2$ IS 1-1

Eg: find the inverse of $f(x) = x^2$ for $x \geq 0$

1. graphically
2. and algebraically

Graphically:



Algebraically:

$$y = x^2 \text{ \& } x \geq 0$$

$$\Longleftrightarrow$$

$$\sqrt{y} = x$$

so $\boxed{f^{-1}(y) = \sqrt{y}}$

another misc. example

Eg: find the inverse of
$$f(x) = \sqrt[3]{x+1}$$

$$y = \sqrt[3]{x+1}$$
$$\Leftrightarrow$$
$$y^3 = x+1$$
$$\Leftrightarrow$$
$$y^3 - 1 = x$$

---

to output y, input $x = y^3 - 1$

so

$$f^{-1}(y) = y^3 - 1$$

Check

$$f(f^{-1}(y)) = f(y^3-1)$$
$$= \sqrt[3]{(y^3-1)+1}$$
$$= \sqrt[3]{y^3}$$
$$= y \quad \checkmark$$

$$f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x+1}\right)$$
$$= \left(\sqrt[3]{x+1}\right)^3 - 1$$
$$= (x+1) - 1 = x \quad \checkmark$$